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AUTHOR(S):

Giga, Yoshikazu; Yoshida, Zensho

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A Bound for the Pressure Integral in a Plasma Equilibrium

Yoshikazu Giga*

北大・理 儀我 美一

Department of Mathematics, Hokkaido University, Sapporo

Zensho Yoshida

東大・工 吉田 善昭

Department of Nuclear Engineering, University of Tokyo, Tokyo

Abstract. An interpolation inequality for the total variation of the gradient of a composite function has been derived by applying the coarea formula. The interpolation inequality has been applied to the study of a bound for the pressure integral concerning a solution of the Grad-Shafranov equation of plasma equilibrium. A weak formulation of the Grad-Shafranov equation has been given to include singular current profiles.

1. Introduction

A simple but essential question in the fusion plasma research is how large plasma energy can be confined by a given magnitude of plasma current.¹⁻⁷ In a magnetohydrodynamic equilibrium of a plasma, the thermal pressure force ∇p is balanced by the magnetic stress $j \times B$, where B is the magnetic flux density, $j = \nabla \times B / \mu_0$ is the current density in the plasma and μ_0 is the vacuum permeability. The plasma equilibrium equation $\nabla p = j \times B$ thus relates the pressure and the current. We want to estimate the maximum of the total pressure with respect to a fixed total current. Mathematically this problem reduces to an a priori estimate for the pressure integral with respect to a solution of the equilibrium equation with a given magnitude of current.

Here we assume a simple two dimensional plasma equilibrium. Let $\Omega \subset \mathbf{R}^2$ be a bounded domain. We consider an infinitely long plasma column; Ω corresponds to

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the cross section of a column containing the plasma. If there is no longitudinal magnetic field, the equilibrium equations are

$$-\Delta\psi = P'(\psi) \quad \text{in } \Omega, \quad (1.1)$$

$$\psi = c \quad \text{on } \partial\Omega, \quad (1.2)$$

$$\int_{\Omega} (-\Delta\psi) dx = \mu_0 I, \quad (1.3)$$

where ψ is the flux function, $P = \mu_0 p$, $P(t)$ is a nonnegative function from \mathbf{R} to \mathbf{R} , $P' = dP(t)/dt$, I is a given positive constant and c is an unknown constant. We assume $P' \geq 0$. Since $-\Delta\psi/\mu_0$ parallels the current density, I represents the total plasma current.

In this paper we study a bound for the total variation of the gradient of $P(\psi)$ in Ω . A crucial step is to establish an interpolation inequality to estimate the total variation of the gradient of $P(\psi)$ in Ω . Our estimate reads

$$\int_{\Omega} |\nabla P(\psi(x))| dx \leq 2 \left(P_{\max} \int_{\Omega} -\Delta\psi dx \right)^{1/2} \left(\int_{\Omega} P'(\psi(x)) dx \right)^{1/2} \quad (1.4)$$

provided that $-\Delta\psi \geq 0$ in Ω and $\psi = c$ on $\partial\Omega$, and that $P' \geq 0$ with $P(c) = 0$, where c is a constant and P_{\max} is the maximum of $P(\psi)$ over Ω . We prove this estimate by using the coarea formula.^{8,9} In section 2 we prove (1.4) and extend it for discontinuous P . In this case the meaning of the equation $-\Delta\psi = P'(\psi)$ is not clear. We shall give a meaning for discontinuous P in section 3.

2. An interpolation inequality

Our goal in this section is to estimate the total variation of $\nabla(P(\psi))$ (as a vector-valued measure), where P is monotone and $-\Delta\psi \geq 0$. We first derive the estimate for smooth ψ .

Theorem 2.1. *Let Ω be a bounded domain in \mathbf{R}^n and c be a constant. Suppose that $P \in C^1(\mathbf{R})$ with $P' \geq 0$ and $P(c) = 0$, and that $\psi \in C^m(\Omega) \cap C^0(\overline{\Omega})$ with*

$$-\Delta\psi \geq 0 \quad \text{in } \Omega, \quad (2.1)$$

$$\psi = c \quad \text{on } \partial\Omega,$$

where $m \geq 2$ and $m \geq n$. Let P_{\max} denote

$$P_{\max} = \sup_{x \in \Omega} P(\psi(x)). \quad (2.2)$$

Then

$$\int_{\Omega} |\nabla P(\psi(x))| dx \leq 2 \left(P_{\max} \int_{\Omega} (-\Delta\psi) dx \right)^{1/2} \left(\int_{\Omega} P'(\psi(x)) dx \right)^{1/2}. \quad (2.3)$$

Proof. If $-\Delta\psi \equiv 0$, then $\psi \equiv c$ on Ω , so (2.3) holds with zero for both sides. If $P'(\psi) \equiv 0$ on Ω or $P_{\max} = 0$, then either $\psi \equiv c$ or $P \equiv 0$. Again (2.3) holds in this case, so we may assume that both integrals in the right hand side of (2.3) is nonzero. We may also assume that the L^1 norm of $-\Delta\psi$ is finite.

For $K > 0$ denote the set of $x \in \Omega$ for which $|\nabla\psi(x)| > K$ by D . Let E denote the complement of D in Ω . From the definition it follows that

$$\begin{aligned} \int_E |\nabla P(\psi(x))| dx &= \int_E P'(\psi) |\nabla\psi| dx \\ &\leq K \int_E P'(\psi) dx \leq K \int_{\Omega} P'(\psi) dx, \end{aligned} \quad (2.4)$$

since $P' \geq 0$.

By the maximum principle to (2.1), we observe that $\psi \geq c$ on Ω so $0 = P(c) \leq P(\psi) \leq P_{\max}$ on Ω . Applying the coarea formula (see e.g. Ref. 8 and 9) yields

$$\int_D |\nabla P(\psi)| dx = \int_{-\infty}^{+\infty} \mathfrak{H}^{n-1}(S_t) P'(t) dt = \int_c^{\psi_{\max}} \mathfrak{H}^{n-1}(S_t) P'(t) dt \quad (2.5)$$

with

$$S_t = D \cap L_t, \quad L_t = \{ x \in \Omega; \psi(x) = t \}, \quad \psi_{\max} = \sup_{x \in \Omega} \psi(x),$$

where \mathcal{H}^{n-1} denotes the $n-1$ dimensional Hausdorff measure. Since $|\nabla \psi| > K$ on D it follows that

$$\begin{aligned} \mathcal{H}^{n-1}(S_t) &= \int_{S_t} |\nabla \psi| |\nabla \psi|^{-1} d\mathcal{H}^{n-1} \\ &\leq K^{-1} \int_{L_t} |\nabla \psi| d\mathcal{H}^{n-1}. \end{aligned}$$

Since $\psi \in C^n(\Omega)$, Sard's theorem¹⁰ implies that L_t is C^n submanifold in Ω for almost every t (a.e. t). Note that $\psi > c$ in Ω and $\psi = c$ on $\partial\Omega$. Thus for $U_t = \{x \in \Omega; \psi(x) > t\}$ we observe $\overline{U_t} \subset \Omega$ for $t > c$. For a.e. $t > c$, L_t is C^n boundary of U_t . Since L_t is t -level set of ψ , $n = \nabla \psi / |\nabla \psi|$ is a unit normal vector field. Applying Green's formula yields

$$\int_{L_t} |\nabla \psi| d\mathcal{H}^{n-1} = \int_{L_t} \nabla \psi \cdot n d\mathcal{H}^{n-1} = \int_{U_t} (-\Delta \psi) dx, \quad t > c.$$

From $-\Delta \psi \geq 0$ it now follows that

$$\int_{L_t} |\nabla \psi| d\mathcal{H}^{n-1} \leq \int_{\Omega} (-\Delta \psi) dx.$$

Wrapping up these two estimates we obtain

$$\mathcal{H}^{n-1}(S_t) \leq K^{-1} \int_{\Omega} (-\Delta \psi) dx.$$

Applying this estimate to (2.5) yields

$$\int_D |\nabla P(\psi)| dx \leq K^{-1} P_{max} \int_{\Omega} (-\Delta \psi) dx, \quad (2.6)$$

where P_{max} is defined in (2.2). Summing (2.4) and (2.6) we obtain

$$\int_{\Omega} |\nabla P(\psi)| dx \leq K \int_{\Omega} P'(\psi) dx + K^{-1} P_{max} \int_{\Omega} (-\Delta \psi) dx \quad (2.7)$$

for arbitrary $K > 0$. Taking

$$K = \left[P_{max} \int_{\Omega} (-\Delta \psi) dx / \int_{\Omega} P'(\psi) dx \right]^{1/2}$$

in (2.7) yields (2.3).

Q.E.D.

If ψ is not C^2 , one should interpret $-\Delta \psi \geq 0$ in the distribution sense. As well known¹¹ a nonnegative distribution is a nonnegative Radon measure. Let μ be a finite Radon measure on a bounded domain Ω in \mathbb{R}^n . The unique solvability of the Dirichlet problem

$$-\Delta \psi = \mu \quad \text{in } \Omega, \quad (2.8a)$$

$$\psi = c \quad \text{on } \partial\Omega \quad (c: \text{constant}) \quad (2.8b)$$

is now well known for smooth boundary $\partial\Omega$. We solve this problem by using a result of Simader¹² when the boundary is C^1 . Let $W^{1,q}(\Omega)$ denote the L^q Sobolev space of order one ($1 < q < \infty$). Let $W_0^{1,q}(\Omega)$ be the subspace $\{u \in W^{1,q}(\Omega); u = 0 \text{ on } \partial\Omega\}$. We denote by $W^{-1,q}(\Omega)$ the dual space of $W_0^{1,q}(\Omega)$ where $1/q = 1 - 1/q'$.

Lemma 2.2 (Theorem 4.6 of Simader¹²). *Let Ω be a bounded domain with C^1 boundary in \mathbb{R}^n . Assume that $1 < q < \infty$. For each $f \in W^{-1,q}(\Omega)$ there is a unique solution $\Phi \in W_0^{1,q}(\Omega)$ for $-\Delta \Phi = f$ in Ω . Moreover the mapping from f to Φ is bounded linear from $W_0^{1,q}(\Omega)$ to $W^{-1,q}(\Omega)$, i.e.,*

$$\|\Phi\|_{1,q} \leq C \|f\|_{-1,q} \quad (2.9)$$

with a constant $C = C(\Omega, q, n)$.

Corollary 2.3. *Let Ω be a bounded domain with C^1 boundary in \mathbb{R}^n . For a finite Radon measure μ on Ω there is a unique solution ψ of (2.8a, b) such that $\psi \in W^{1,r}(\Omega)$ for $1 < r < n/(n-1)$.*

Proof. Observe that $r' > n$ implies $W_0^{1,r'}(\Omega) \subset \overline{C(\Omega)}$ by the Sobolev inequality. This yields $\mu \in W^{-1,r}(\Omega)$ by a duality, where $1/r = 1 - 1/r'$. Applying Lemma 2.2 with $f = \mu$ obtains a unique solution ψ by $\psi = \Phi + c$.

Q.E.D.

Theorem 2.4. *Let Ω be a bounded domain with C^1 boundary in \mathbb{R}^n . Let c be a constant. Suppose that $P \in C^1(\mathbb{R})$ with $P' \geq 0$ and $P(c) = 0$. Suppose that $\psi \in W^{1,r}(\Omega)$ for some r such that $1 < r < n/(n-1)$, and that ψ satisfies*

$$\begin{aligned} -\Delta\psi &\geq 0 \quad \text{in } \Omega \text{ (in the distribution sense),} \\ \psi &= c \quad \text{on } \partial\Omega. \end{aligned}$$

Let ψ_{\max} be the essential supremum of ψ over Ω . Assume that P and P' are bounded on $[c, \psi_{\max})$. Then

$$\int_{\Omega} |\nabla P(\psi(x))| dx \leq 2 (P_{\max} \|\Delta\psi\|_1)^{1/2} \left(\int_{\Omega} P'(\psi(x)) dx \right)^{1/2}, \quad (2.10)$$

where $P_{\max} = \sup \{P(\sigma); c \leq \sigma \leq \psi_{\max}\}$ and $\|\cdot\|_1$ denotes the total variation of a measure on Ω .

For the proof of this Theorem, the reader is referred to Ref. 13.

We next extend the inequality (2.9) when a nondecreasing function P is not necessarily continuous. Let us give an interpretation of each integral appeared in (2.9). Instead of the integral $\int_{\Omega} P'(\psi) dx$, we consider

$$[P'(\psi)] = \inf \lim_{l \rightarrow \infty} \int_{\Omega} P'_l(\psi) dx.$$

Here the infimum is taken over all sequence $P_l \in C^1(\mathbf{R})$ with $P'_l \geq 0$ such that $P_l(\psi) \rightarrow P(\psi)$ in $L^s(\Omega)$ for some $1 \leq s < \infty$ as $l \rightarrow \infty$ and that $(P_l)_{\max} \rightarrow \text{ess sup}_{\Omega} P(\psi)$. We say $\{P_l\}$ is an admissible approximation of P if these properties hold. If P is itself C^1 and satisfies the assumptions in Theorem 2.4, P itself is an admissible approximation so for such a P we have

$$[P'(\psi)] \leq \int_{\Omega} P'(\psi) dx.$$

Since $\int_{\Omega} |\nabla P(\psi)| dx$ is the total variation of $\nabla P(\psi)$ on Ω , i.e.

$$\begin{aligned} \|\nabla P(\psi)\|_1 &= \int_{\Omega} |\nabla P(\psi(x))| dx \\ &:= \sup \left\{ \int_{\Omega} P(\psi(x)) \nabla \cdot \phi(x) dx ; \phi \in C_0^1(\Omega), |\phi(x)| \leq 1 \text{ on } \Omega \right\}, \end{aligned}$$

it is easy to see

$$\|\nabla P(\psi)\|_1 \leq \lim_{l \rightarrow \infty} \int_{\Omega} |\nabla P_l(\psi)| dx$$

for any admissible approximation $\{P_l\}$ of P since $\sup \lim \leq \lim \sup$. We have thus proved the following assertion.

Theorem 2.5. *Assume the hypotheses of Theorem 2.4 concerning c , Ω and ψ . Let P be a nondecreasing function on \mathbf{R} with $P(c) = 0$. Then*

$$\|\nabla P(\psi)\|_1 \leq 2 (P_{\max} \|\Delta \psi\|_1)^{1/2} [P'(\psi)]^{1/2} \quad (2.11)$$

provided that $P_{\max} = \text{ess sup}_{\Omega} P(\psi)$ is finite.

Remark 2.6. If $P(\sigma) = \sigma$, the inequality (2.10) is an interpolation inequality

$$\|\nabla \psi\|_1 \leq 2 (P_{\max} \|\Delta \psi\|_1)^{1/2} |\Omega|^{1/2},$$

where $|\Omega|$ denotes the Lebesgue measure of Ω .

3. Weak solution of the Grad-Shafranov equation

We shall give a meaning of $-\Delta \psi = P'(\psi)$ when a nondecreasing function P is not continuous and ψ is not smooth.

Definition 3.1. Suppose that $\psi \in W^{1,r}(\Omega)$ for some r , $1 < r < \infty$ and that P is nondecreasing. We say ψ and P satisfy

$$-\Delta \psi = P'(\psi) \text{ in } \Omega$$

if the following properties hold.

- (i) $-\Delta \psi \geq 0$ on Ω in the distribution sense.
- (ii) There is an admissible sequence $\{P_l\}$ such that

$$\lim_{l \rightarrow \infty} \int_{\Omega} (-\Delta \psi - P'_l(\psi)) \varphi \, dx = 0$$

for all $\varphi \in \overline{C(\Omega)}$.

Theorem 3.2. Let Ω be a bounded domain with C^1 boundary in \mathbb{R}^n . Let c be a constant. Assume that P is a nondecreasing function on \mathbb{R} . Assume that $\psi \in W^{1,r}(\Omega)$ for some r , $1 < r < n/(n-1)$ and that ψ satisfies

$$-\Delta \psi = P'(\psi) \text{ in } \Omega \text{ (in the sense of Definition 3.1)}$$

$$\psi = c \quad \text{on } \partial\Omega.$$

Then

$$\|\nabla P(\psi)\|_1 \leq 2 P_{max}^{1/2} \mu_0 I, \quad (3.1)$$

where

$$I = \mu_0^{-1} \int_{\Omega} (-\Delta \psi) dx = \mu_0^{-1} \|\Delta \psi\|_1.$$

Proof. We may assume $P_{max} < \infty$. By Definition 3.1 (ii) with $\phi \equiv 1$ we observe that

$$[P'(\psi)] \leq \lim_{l \rightarrow \infty} \int_{\Omega} P_l'(\psi) dx = \int_{\Omega} (-\Delta \psi) dx = \|\Delta \psi\|_1$$

since $-\Delta \psi \geq 0$. The inequality (2.11) yields (3.1).

Q.E.D.

4. Discussions

In plasma physics, the poloidal beta ratio, which is define by

$$\beta = \int_{\Omega} p dx / (I^2 \mu_0 / 8\pi) = 8\pi \int_{\Omega} P(\psi) dx / \left(\int_{\Omega} (-\Delta \psi) dx \right)^2,$$

is an important quantity to characterize a plasma equilibrium. In the case of the space dimension $n = 2$, the Payne-Rayner inequality¹⁴ applies to the estimate of β , and one finds $\beta \leq 1$. A general toroidal equilibrium problem includes two different effects; In the equilibrium equation (1.1), $-\Delta \psi$ should be replaced by a more complicated term including the toroidal curvature effect, and a new term should be added on the right-hand side, which represents the diamagnetic effect of the longitudinal magnetic field. Limitation of β in such a situation has been discussed by many authors, while no rigorous estimate of the bound have been given. Extension of the Payne-Rayner inequality will be discussed elsewhere to estimate the bound for β .

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